A variational coupling for a
totally asymmetric exclusion process
with long jumps but no passing

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Abstract

We prove a weak law of large numbers for a tagged particle in a totally asymmetric exclusion process on the one-dimensional lattice. The particles are allowed to take long jumps but not pass each other. The object of the paper is to illustrate a special technique for proving such theorems. The method uses a coupling that mimics the Hopf-Lax formula from the theory of viscosity solutions of Hamilton-Jacobi equations.

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1 Introduction

The purpose of this paper is to explain a technique for deriving scaling limits that applies to certain totally asymmetric particle processes. The example we use to illustrate the method is a one-dimensional exclusion process whose particles are allowed to take arbitrarily long jumps, as long as particles do not pass each other. Thus the ordering of particles is preserved, and the process models traffic on a single-lane highway. To the author's knowledge, the hydrodynamics of this particular process have not been addressed before.

   The key point of the technique is this: There exists a coupling that expresses a process with general initial conditions in terms of a family of processes that have a simple structure. The evolution from a general initial configuration is constructed from the simple cases by a variational formula, hence the term "variational coupling". The benefit of the scheme is that the limit behavior of the simple processes is tractable, and through the coupling one obtains results for the general process.

   This paper is intended as an introduction to the technique. Fairly complete proofs are provided, and we have not strived to obtain the best possible result for the process under study. Some extensions of the theorem of the paper and shortcomings of the method are addressed in Section 6. A review of limit theorems for tagged particles and references to literature can be found in [3].

   Notational remarks. For a real number x, $[x]$ denotes the largest integer less than or equal to x. $\mathbf{Z}_-$ is the set of nonpositive integers $\{\ldots, -3, -2, -1, 0\}$.

2 Description and construction of the process

The process consists of particles that move on the one-dimensional lattice $\mathbf{Z}$. At most one particle is permitted on each site. (This is the meaning of the name "exclusion process".) We label the particles by integers $i \in \mathbf{Z}$. The state of the process is a sequence of integers $\sigma = (\sigma(i) : i \in \mathbf{Z})$, where $\sigma(i)$ is the location of particle i. We shall also use "$\sigma(i)$" as the name of the particle, because typically we need to discuss simultaneously several different processes denoted by different Greek letters. The dynamics will be such that the ordering of particles is preserved. So we can assume that

$$\sigma(i) + 1 \leq \sigma(i+1) \text{ for all } i \in \mathbf{Z}. \tag{1}$$

This condition contains both the exclusion rule and the no-passing rule.

It is also convenient to allow the values $\pm\infty$ for $\sigma(i)$. Assumption (1) remains valid. If $\sigma(i) < \infty = \sigma(i+1)$, then $\sigma(i)$ is the rightmost particle on the lattice $\mathbf{Z}$ because $\sigma(j) = \infty$ for all $j > i$. Similarly there is a leftmost particle if some $\sigma(j) = -\infty$.

To summarize, the state space of the process is the set of $\mathbf{Z}\cup\{\pm\infty\}$-valued sequences σ that satisfy (1).

For the dynamics, assume given a sequence β_k, $k \geq 1$, of nonnegative real numbers. These are the rates. For each k and each i, particle $\sigma(i)$ independently attempts to jump k steps to the right at exponential rate β_k. If there are k empty lattice spaces in front of $\sigma(i)$, this jump can be executed. If there are not, in other words, if $\sigma(i+1) \leq \sigma(i) + k$, then $\sigma(i)$ jumps as much as it can, to the location $\sigma(i+1) - 1$ right behind the next particle. The jumps occur independently for all particles $\sigma(i)$ and all jump sizes k.

For nonnegative integers m, define

$$B_m = \sum_{k=1}^{\infty} k^m \beta_k .\tag{2}$$

We assume

$$B_2 < \infty.\tag{3}$$

Then also $B_0 < \infty$, which guarantees that there are finitely many jump attempts in finite time intervals almost surely. $B_2 < \infty$ implies that the distance traveled by any particle during a finite time interval has a finite second moment.

To rigorously construct the process, let $\{\mathcal{D}_i : i \in \mathbf{Z}\}$ be an i.i.d. collection of homogeneous rate B_0 Poisson point processes on $[0, \infty)$. In other words, a time interval $[t, t + h)$ has a Poisson($B_0 h$)-distributed random number of points (or *epochs*) from the process $\mathcal{D}_i$. For each process $\mathcal{D}_i$, label the points independently by integers $k \geq 1$, with probability β_k/B_0 for label k. Let $\mathcal{D}_i^k$ denote the point process of k-epochs (that is, the points with label k) from process $\mathcal{D}_i$. It follows that $\{\mathcal{D}_i^k : i \in \mathbf{Z}, k \geq 1\}$ is a collection of mutually independent homogeneous Poisson point processes on $[0, \infty)$, and the rate of $\mathcal{D}_i^k$ is β_k. If $\beta_k = 0$ the point process $\mathcal{D}_i^k$ is empty for all i.

The idea of the construction is that particle $\sigma(i)$ reads its jump commands from the Poisson process $\mathcal{D}_i$, and attempts to execute a k-step jump at each epoch of $\mathcal{D}_i^k$. Let $\sigma(i, t)$ denote the location of particle $\sigma(i)$ at time $t \geq 0$, and

$\sigma(t) = (\sigma(i,t) : i \in \mathbf{Z})$ the entire state at time t. Assume that a deterministic initial configuration $\sigma(0) = (\sigma(i,0) : i \in \mathbf{Z})$ has been specified. Informally, the rule of evolution is this:

Jump rule. Suppose τ is an epoch of $\mathcal{D}_i^k$. Assume that the state $\sigma(t)$ has been defined for $t < \tau$. At time τ set

$$\sigma(i,\tau) = \min \{\sigma(i,\tau-) + k \,, \; \sigma(i+1,\tau-) - 1\} \, . \tag{4}$$

After τ, $\sigma(i)$ stays constant until the next epoch in $\mathcal{D}_i$. The same scheme happens simultaneously and independently for all particles $\sigma(i)$.

Since there are infinitely many particles, infinitely many jumps are attempted almost surely in any time interval $(0, \varepsilon)$. In particular, there cannot be a first jump. Consequently some argument is needed to justify the induction in the jump rule, so that formula (4) can be used to define the successive locations of particle $\sigma(i)$.

The realizations $\{\mathcal{D}_i^k\}$ for which we can construct the dynamics satisfy these assumptions:

(a) Each $\mathcal{D}_i$ has only finitely many epochs in each bounded

  time interval.

(b) There are no simultaneous jump attempts.

(c) There are arbitrarily large i_0 and t_0 such that $\mathcal{D}_{-i_0}$ and $\mathcal{D}_{i_0}$

  have no epochs in $[0, t_0]$. $\tag{5}$

It is a standard fact of Poisson processes that these assumptions are satisfied by almost every realization $\{\mathcal{D}_i^k\}$.

Assumptions (5) allow us to prove that the jump rule defines the evolution unambiguously for all particles $\sigma(i)$ and all times $0 \leq t < \infty$: Given i and t, part (c) of (5) gives i_0 and t_0 so that $-i_0 < i < i_0$ and $t_0 > t$, and so that $\mathcal{D}_{-i_0}$ and $\mathcal{D}_{i_0}$ have no epochs in $[0, t_0]$. Consequently $\sigma(-i_0, t) = \sigma(-i_0, 0)$ and $\sigma(i_0, t) = \sigma(i_0, 0)$ for all $t \in [0, t_0]$, and the evolution of particles $\sigma(j)$ for $-i_0 < j < i_0$ is isolated from the rest of the process up to time t_0. By parts (a) and (b) of (5), the locations $\sigma(j, t)$ for $-i_0 < j < i_0$ and $t \in [0, t_0]$ can be computed by applying the jump rule to the finitely many potential

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jump times in $\cup_{-i_0 < j < i_0} \mathcal{D}_j \cap [0, t_0]$, in their temporal order. In particular, the motion of particle $\sigma(i)$ is defined up to time t.

This way the evolution $\sigma(\cdot) = \{\sigma(t) : t \geq 0\}$ is defined as a function of an arbitrary initial configuration $\sigma(0)$, and of almost every realization of the jump times $\{\mathcal{D}_i^k\}$. If $(\Omega, \mathcal{F}, P)$ is a probability space on which are defined the point processes $\{\mathcal{D}_i^k\}$ and a random initial configuration $\sigma(0)$, then $\sigma(\cdot)$ is a stochastic process defined on this same probability space. Note: The Poisson point processes have to be independent of the initial configuration $\sigma(0)$.

Constructing particle systems from Poisson processes of potential jump times is a standard procedure, and can be found in general references on particle systems such as [1, 4, 5]. A useful property of this construction is that it preserves orderings between two processes:

Lemma 1 *Suppose the probability space $(\Omega, \mathcal{F}, P)$ contains two initial configurations $\sigma'(\cdot, 0)$ and $\sigma''(\cdot, 0)$ that satisfy $\sigma'(i, 0) \leq \sigma''(i, 0)$ for all $i \in \mathbf{Z}$ with probability 1. If the processes $\sigma'(\cdot, t)$ and $\sigma''(\cdot, t)$ obey the same point processes $\{\mathcal{D}_i^k\}$, we have $\sigma'(i, t) \leq \sigma''(i, t)$ for all $i \in \mathbf{Z}$ and $t \geq 0$, with probability 1.*

Proof. Choose i_0 and t_0 to satisfy part (c) of assumption (5). We shall prove that the ordering $\sigma'(i, t) \leq \sigma''(i, t)$ holds for $-i_0 \leq i \leq i_0$ and $t \leq t_0$. It holds for $i = \pm i_0$ because particles $\sigma'(\pm i_0)$ and $\sigma''(\pm i_0)$ do not attempt to jump during the time interval $[0, t_0]$. By parts (a) and (b) of (5), we may do induction over the jump times. Suppose $\tau \in \mathcal{D}_i^k$ is the first epoch at which the ordering is violated, for $-i_0 \leq i \leq i_0$ and $\tau \leq t_0$. Then we must have $\sigma'(i, \tau) > \sigma''(i, \tau)$ but $\sigma'(i, \tau-) \leq \sigma''(i, \tau-)$. Since $\sigma''(i)$ was not allowed to jump as far as $\sigma'(i)$ even though they both attempted a k-jump, it must be that $\sigma'(i + 1, \tau-) > \sigma''(i + 1, \tau-)$. Since τ is the first time the ordering is violated for indices $-i_0 \leq i \leq i_0$, it must be that $i + 1 > i_0$. Since $i_0 \geq i$, we conclude that $i = i_0$. But $\mathcal{D}_{i_0}$ was chosen to have no epochs up to time t_0. We have reached a contradiction, so there can be no first violation of the ordering for $-i_0 \leq i \leq i_0$ and $t \leq t_0$. Since i_0 and t_0 can be taken arbitrarily large, the proof is complete. ∎

3   The heuristic behind the method

We are interested in a macroscopic view of the process. By this we mean its large scale behavior, distinct from the local interactions of the individual particles $\sigma(i)$ that represent the microscopic dynamics. The ratio between macroscopic and microscopic scales is given by a parameter n, so that macroscopic space and time units represent n microscopic space and time units. We hope that a deterministic macroscopic description of the dynamics emerges in the limit $n \to \infty$, where the macroscopic and microscopic scales become infinitely separated.

What form should the deterministic macroscopic description take? Suppose we have a function $u(x,t)$ defined for macroscopic space-time variables (x,t) such that the approximate equality

$$u(x,t) \approx n^{-1}\sigma([nx], nt) \tag{6}$$

holds with high probability for large n. Since the rate of advance of particle $\sigma([nx], nt)$ depends only on the distance to the next particle on the right, we might expect this same structure also at the macroscopic level, so that

$$u_t = f(u_x) \tag{7}$$

for some function f. Equation (7) is a first-order partial differential equation of the Hamilton-Jacobi type. It is usually written as $u_t - f(u_x) = 0$, and then the function $-f$ is known as the *Hamiltonian*.

The exclusion rule and (6) force $u_x \geq 1$, so f is a function defined on $[1, \infty)$. If the distance to the next particle ahead is one (no empty sites) a particle cannot jump, and hence $f(1) = 0$. The rate of advancement cannot decrease as the distance to the next particle increases, so f ought to be nondecreasing. No particle can advance faster than at rate B_1, so $f \leq B_1$. Since f is nondecreasing and bounded above, it appears reasonable to assume that f is concave.

Thus, assuming our heuristic reasoning is reliable, if there is to be a law of large numbers

$$n^{-1}\sigma([nx], nt) \to u(x,t), \tag{8}$$

then u should satisfy equation (7) with a convex Hamiltonian $-f$. There is a well-known formula for solving (7) when the Hamiltonian is convex: Let

$$g(x) = \sup_{v \geq 1}\{vx + f(v)\} \tag{9}$$

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be the convex conjugate of $-f$. Suppose the initial data for (7) is given by a function u_0 on $\mathbf{R}$. Then the *Hopf-Lax formula*

$$u(x,t) = \inf_{y \geq x} \left\{ u_0(y) + tg\left(\frac{x-y}{t}\right) \right\} \tag{10}$$

defines the unique viscosity solution of (7) with initial data $u(x,0) = u_0(x)$ (Theorem 3, Section 10.3 in [2]). Note: Since $f \geq 0$, (9) shows that $g(x) = \infty$ for $x > 0$. That is why we may restrict the infimum in (10) to $y \geq x$.

However, our task is not to solve the differential equation (7) with a given f. In some sense we already have the solutions, at least in their microscopic form $\sigma([nx], nt)$ because we already constructed the particle process. Instead, our problem is to prove that the process σ admits a macrosopic description of the type (7), and to find f and g, if possible. The Hopf-Lax formula suggests a line of attack on our problem.

Suppose we are allowed to feed a single input $u_0 = \varphi_0$ to the system described by (10), and measure the output $\varphi(x,t) = u(x,t)$. Can we obtain g? Take

$$\varphi_0(y) = \begin{cases} \infty & \text{if } y > 0, \\ y & \text{if } y \leq 0. \end{cases} \tag{11}$$

With $u_0 = \varphi_0$, differentiating inside the braces in (10) shows that $\varphi(x,t) = u(x,t) = tg(x/t)$. (The duality (9) implies that the slope of g is everywhere at least 1, since this is the lower bound for v.) Once we have g, we can compute the solutions to (7) by the Hopf-Lax formula (10). In other words, the single evolution $\varphi(x,t)$ started with (11) contains the information for constructing *all* evolutions from their initial data.

To understand this point better, let u_0 be an arbitrary initial function for the evolution, and consider this family of special initial data, indexed by $z \in \mathbf{R}$:

$$\varphi_0^z(y) = u_0(z) + \varphi_0(y), \quad y \in \mathbf{R}. \tag{12}$$

These are simply translates of the initial data (11). Find the evolution from (10):

$$\begin{aligned}
\varphi^z(x,t) &= \inf_{y \geq x} \left\{ \varphi_0^z(y) + tg\left(\frac{x-y}{t}\right) \right\} \\
&= u_0(z) + \inf_{y \geq x} \left\{ \varphi_0(y) + tg\left(\frac{x-y}{t}\right) \right\} \\
&= u_0(z) + tg(x/t).
\end{aligned}$$

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We substitute the evolution $\varphi^z(x,t)$ into (10) to obtain

$$u(x,t) = \inf_{y \geq x} \varphi^y(x-y,t)\,. \tag{13}$$

Equations (12) and (13) contain the message: A general evolution $u(x,t)$ starting from initial data u_0 can be computed from the family of solutions $\{\varphi^y : y \in \mathbf{R}\}$, whose initial data are translates $\varphi_0^y = u_0(y) + \varphi_0$ of the function φ_0.

Let us move the discussion to the microscopic particle level. We can realize the initial data (11) microscopically, in the sense of the limit (8), by a natural particle configuration:

$$\sigma(i,0) = \begin{cases} \infty & \text{if } i > 0, \\ i & \text{if } i \leq 0. \end{cases} \tag{14}$$

This says that the lattice is empty to the right of the origin, while to the left all sites are occupied, the origin itself included. Since (14) is a microscopic version of (11), we would expect that the process $\sigma(t)$ with initial configuration (14) is a microscopic version of the solution $\varphi(x,t) = tg(x/t)$. Translates of these solutions suffice to express the most general solution, as observed in (13).

The key question becomes: Is there a microscopic principle that corresponds to (13) and expresses a process with a general initial configuration in terms of a family of processes of the type that start from (14)? The construction we seek is the coupling we now turn to.

4 The coupling

Assume given an arbitrary deterministic initial particle configuration $\sigma(0) = (\sigma(j,0) : j \in \mathbf{Z})$, and let $\sigma(j,t)$ denote the process constructed in terms of the Poisson point processes $\{\mathcal{D}_j^k\}$ as explained in Section 2. For each finite initial location $\sigma(j,0)$ we construct an auxiliary exclusion process $\zeta^j = (\zeta^j(i,t) : i \in \mathbf{Z}_-, t \geq 0)$. Initially

$$\zeta^j(i,0) = \sigma(j,0) + i \text{ for } i \in \mathbf{Z}_-. \tag{15}$$

The particles $\zeta^j(i)$ for $i > 0$ are not needed. According to our convention we may think that they reside permanently at ∞. Thus ζ^j is a process that is a spatial translate of the process started from (14).

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For each process ζ^j, the dynamical description is the same as for σ: Jumps of size k are attempted independently at rate β_k. Each jump is carried as far as possible without violating the exclusion and no-passing rules. Each process ζ^j uses the same collection of Poisson processes $\{\mathcal{D}_i^k\}$ as does σ, but with a translation of the index:

Jump rule for process ζ^j. At each epoch τ of $\mathcal{D}_{i+j}^k$, set

$$\zeta^j(i,\tau) = \min\left\{\zeta^j(i,\tau-) + k \,,\, \zeta^j(i+1,\tau-) - 1\right\}. \tag{16}$$

Between epochs of $\mathcal{D}_{i+j}$, $\zeta^j(i)$ stays constant.

Subject to this rule, the processes $\{\zeta^j\}$ are constructed exactly as σ. The processes σ and $\{\zeta^j\}$ are all defined on the same probability space $(\Omega, \mathcal{F}, P)$ of the Poisson jump times. If this probability space also supports a random initial configuration $\sigma(0)$, then the processes $\{\zeta^j\}$ are functions of both $\sigma(0)$ and $\{\mathcal{D}_i^k\}$, through the translations (15) and the jump rule.

This arrangement where several processes are defined on a common probability space to facilitate their direct comparison is known as a *coupling*. It should be evident that the processes are invisible to each other, i.e. particles of one process do not interfere with the jumps of the other processes.

The key property of this coupling is the microscopic counterpart of (13):

Lemma 2 *The equality*

$$\sigma(i,t) = \inf_{j:j\geq i} \zeta^j(i-j,t) \tag{17}$$

holds for all $i \in \mathbf{Z}$ and $t \geq 0$, almost surely.

Proof. The exclusion rule (1) and (15) imply that (17) holds at time 0. The jump rules ensure that for each $j \geq i$, particles $\zeta^j(i-j)$ and $\sigma(i)$ attempt k-jumps at common time points, namely at the epochs of $\mathcal{D}_i^k$. Thus for a fixed i the validity of (17) can change only at epochs of $\mathcal{D}_i$.

Fix i_1 and t_1. To prove that (17) holds for $i = i_1$ up to time t_1, use part (c) of assumption (5) to find i_0 such that $-i_0 < i_1 < i_0$, and so that $\mathcal{D}_{-i_0}$ and $\mathcal{D}_{i_0}$ have no epochs in the time interval $[0,t_1]$. By parts (a) and (b) of (5), we can do induction on the finitely many epochs in $\cup_{-i_0 < i < i_0} \mathcal{D}_i \cap [0,t_1]$.

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So suppose τ is an epoch of $\mathcal{D}_i^k$ for some $k \geq 1$ and some i between $-i_0$ and i_0, and assume that (17) holds for all i between $-i_0$ and i_0 and all $t < \tau$.

First we show that $\leq$ holds in (17) right after the jump epoch τ. Let $j \geq i$ be arbitrary. By the induction assumption, $\zeta^j(i - j + 1, \tau-) \geq \sigma(i+1, \tau-)$, and consequently $\zeta^j(i - j)$ can jump at least as far as $\sigma(i)$ can. Since before the jump we have $\zeta^j(i - j, \tau-) \geq \sigma(i, \tau-)$, and both particles attempt a k-jump, the inequality $\zeta^j(i - j, \tau) \geq \sigma(i, \tau)$ holds also at time τ after the jump has been completed. The same argument works for all $j \geq i$.

Next we show that right after τ there is some $j \geq i$ that satisfies $\zeta^j(i - j, \tau) = \sigma(i, \tau)$. If $\sigma(i)$ was allowed to jump the full k steps, any j such that $\zeta^j(i - j, \tau-) = \sigma(i, \tau-)$ will do, and by induction there is some such j.

Suppose on the contrary that $\sigma(i)$ was at least partially blocked by $\sigma(i+1)$ at time $\tau-$, so after the jump we have $\sigma(i, \tau) = \sigma(i+1, \tau) - 1$. By induction, we can pick $j \geq i+1$ such that $\zeta^j(i - j + 1, \tau-) = \sigma(i+1, \tau-)$. By the earlier part of the proof, $\zeta^j(i - j, \tau) \geq \sigma(i, \tau)$. On the other hand, the exclusion rule for the process ζ^j stipulates that $\zeta^j(i - j, \tau) \leq \zeta^j(i - j + 1, \tau) - 1 = \sigma(i+1, \tau) - 1 = \sigma(i, \tau)$. (Notice that by part (b) of assumption (5) there is no epoch τ in $\mathcal{D}_{i+1}$.) This shows that after the jump we necessarily have the equality $\zeta^j(i - j, \tau) = \sigma(i, \tau)$. ∎

By subtracting off the centering at $\sigma(j, 0)$ from ζ^j we obtain a family of processes $\{\xi^j\}$ that are all equal in distribution. Define

$$\xi^j(i, t) = \zeta^j(i, t) - \sigma(j, 0) \text{ for all } i, j, \text{ and } t. \tag{18}$$

Then we can turn the coupling equality (17) into a statement that is a microscopic version of the Hopf-Lax formula (10):

$$\sigma(i, t) = \inf_{j: j \geq i} \left\{ \sigma(j, 0) + \xi^j(i - j, t) \right\}. \tag{19}$$

The processes $\{\xi^j\}$ are identically distributed on account of the requirement that $(\sigma(j, 0))$ and the Poisson processes $\{\mathcal{D}_j^k\}$ are independent. The advantage of (19) over (17) is that the effects of $(\sigma(j, 0))$ and $\{\mathcal{D}_j^k\}$ have been separated in two distinct terms inside the braces.

5   The tagged particle limit

As a corollary of the coupling we obtain the limit discussed heuristically in Section 3. The precise assumptions are as follows: There is a sequence of

processes σ_n constructed as in Section 2, and an increasing function u_0 on $\mathbf{R}$. At time 0, the convergence

$$\lim_{n\to\infty} n^{-1}\sigma_n([ny], 0) = u_0(y) \tag{20}$$

holds in probability for each $y \in \mathbf{R}$. More precisely, if $(\Omega_n, \mathcal{F}_n, P_n)$ is the probability space on which the process σ_n is defined, then

$$\lim_{n\to\infty} P_n\left(|\sigma_n([ny], 0) - nu_0(y)| \geq n\varepsilon\right) = 0 \tag{21}$$

for each $y \in \mathbf{R}$ and $\varepsilon > 0$.

Theorem 1 *Under assumption* (20), *the convergence*

$$\lim_{n\to\infty} n^{-1}\sigma_n([nx], nt) = u(x, t) \tag{22}$$

holds in probability for each $x \in \mathbf{R}$ and $t > 0$. The deterministic limit $u(x, t)$ can be macroscopically defined by

$$u(x, t) = \inf_{y:y\geq x} \left\{ u_0(y) + tg\left(\frac{x-y}{t}\right) \right\}, \tag{23}$$

where g is a certain convex function on $(-\infty, 0]$ determined by the rules of the exclusion process.

The meaning of the limit in (22) is the same as above for (20), namely that

$$\lim_{n\to\infty} P_n\left(|\sigma_n([nx], nt) - nu(x, t)| \geq n\varepsilon\right) = 0 \tag{24}$$

for each $\varepsilon > 0$. By the discussion about Hamilton-Jacobi equations in Section 3, $u(x, t)$ can be equivalently characterized as the unique viscosity solution of $u_t = f(u_x)$ with initial data $u(x, 0) = u_0(x)$, where f is the negative of the convex conjugate of g. The theorem says that in this exclusion system, a tagged particle has a well-defined macroscopic velocity $f(u_x)$ as long as initially the particle distribution is sufficiently regular as specified by assumption (20).

Proof of Theorem 1

To prove Theorem 1, we forge a rigorous connection between the matching microscopic and macroscopic descriptions (19) and (23). For each n, we have the processes ζ_n^j and ξ_n^j, defined as in Section 4 relative to the process σ_n. All the processes ξ_n^j are identical in distribution, so as long as we discuss distributional properties, we may leave out the sub- and superscript n and j. The function g that appears in the theorem is defined by the limit of the next lemma.

Lemma 3 *There exists a finite, continuous, convex function g on $(-\infty, 0]$ such that*

$$\lim_{n \to \infty} n^{-1}\xi([nx], nt) = tg(x/t) \tag{25}$$

in probability, for each $x \le 0$ and $t > 0$.

Proof of Lemma 3. We start by deriving a useful subadditivity. Assume the process $\xi(i, t)$ has been constructed on a probability space $(\Omega, \mathcal{F}, P)$ as explained in Section 2, so that particle $\xi(i)$ gets its jump commands from the labeled Poisson process $\mathcal{D}_i$. To perform comparisons, we explicitly include the particles that are permanently at infinity: $\xi(i, t) = \infty$ for all $i > 0$ and $t \ge 0$.

Fix an integer $h \in \mathbf{Z}_-$ and a time $s > 0$. Construct two new processes σ' and σ'' on the same probability space $(\Omega, \mathcal{F}, P)$ as follows: The initial configurations are

$$\sigma'(i, 0) = \xi(h + i, s) \text{ for all } i \in \mathbf{Z} \tag{26}$$

and

$$\sigma''(i, 0) = \begin{cases} \infty & \text{if } i > 0, \\ \xi(h, s) + i & \text{if } i \le 0. \end{cases} \tag{27}$$

We stipulate that particles $\sigma'(i)$ and $\sigma''(i)$ read their jump commands from $(\mathcal{D}_{h+i} - s) \cap [0, \infty)$. In other words, σ' and σ'' read the Poisson processes $\{\mathcal{D}_i\}$ after first translating the index by h and time by s. By Lemma 1,

$$\sigma'(i, t) \le \sigma''(i, t) \text{ for all } i \in \mathbf{Z} \text{ and } t \ge 0. \tag{28}$$

Since σ' continues ξ with translated index and time, $\sigma'(i, t) = \xi(h + i, s + t)$. For σ'' we can write $\sigma''(i, t) = \xi(h, s) + \widetilde{\xi}(i, t)$ where $\widetilde{\xi}$ is equal in distribution

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to ξ but independent of it. Thus (28) turns into

$$\xi(h + i, s + t) \leq \xi(h, s) + \widetilde{\xi}(i, t),\tag{29}$$

where the random variables on the right are independent.

Suppose first that x in (25) is a nonpositive integer, $x = i$. Let F_n be the distribution function of $\xi(ni, nt)$. Then (29) gives $F_{n+m} \geq F_n * F_m$. By the Kesten-Hammersley lemma from subadditive ergodic theory there exists a function $\gamma(i, t)$ such that

$$\lim_{n \to \infty} n^{-1}\xi(ni, nt) = \gamma(i, t)\tag{30}$$

in probability, for all $i \in \mathbf{Z}_-$ and $t \geq 0$. A proof of this lemma can be found on p. 20 in [11]. The assumption of finite second moments required by this lemma is satisfied by assumption (3).

The rest of the proof consists of extending the limit to all $x \leq 0$, by using regularity and monotonicity properties of the process and $\gamma(i, t)$. It is obvious that $\gamma(i, t)$ is nondecreasing in both i and t. For $i = 0$ we have $\gamma(0, t) = B_1 t$, because B_1 is the speed of an unobstructed particle. Furthermore

$$i \leq \gamma(i, t) \leq i + B_1 t\tag{31}$$

for all $i \in \mathbf{Z}_-$ and $t \geq 0$, because $\xi(i, 0) = i$ and B_1 is the maximal average speed.

Next we argue the t-continuity of $\gamma(i, t)$. During the time interval $(nt, nt + n\varepsilon]$ particle $\xi(ni)$ attempts a Poisson$(B_0 n\varepsilon)$ number of jumps. The attempted jump sizes are i.i.d. random variables $\{X_i\}$ with common distribution $P(X_i = k) = \beta_k/B_0$. Due to the exclusion and no-passing rules, the actual jumps are at most X_i, so we can write

$$\xi(ni, nt + n\varepsilon) \leq \xi(ni, nt) + \sum_{i=1}^{N} X_i\tag{32}$$

where $N \sim$ Poisson$(B_0 n\varepsilon)$. It follows that

$$\gamma(i, t + \varepsilon) \leq \gamma(i, t) + B_1 \varepsilon.\tag{33}$$

In particular, $\gamma(i, t)$ is Lipschitz continuous in t.

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A homogeneity holds: If m is a positive integer, then

$$
\begin{aligned}
\gamma(mi, mt) &= \lim_{n \to \infty} n^{-1} \xi(nmi, nmt) \\
&= m \cdot \lim_{n \to \infty} (nm)^{-1} \xi(nmi, nmt) \\
&= m\gamma(i, t).
\end{aligned}
$$

From this it follows that we can define unambiguously for rational $r \le 0$

$$
\gamma(r, t) = \frac{1}{m}\gamma(mr, mt), \tag{34}
$$

where m is any positive integer such that $mr \in \mathbf{Z}_-$.

We need to prove that the extension (34) makes sense for the process, namely that

$$
\lim_{n \to \infty} n^{-1} \xi([nr], nt) = \gamma(r, t) \tag{35}
$$

for rational $r < 0$ and all $t > 0$. If n runs along the subsequence $n = km$, $k = 1, 2, 3, \dots$, where m is the integer in (34), then (35) is valid. For other values of n we interpolate using the monotonicity of the process. No-passing and total asymmetry of the dynamics imply that

$$
\xi(i, s) \le \xi(j, s) \le \xi(j, t)
$$

for all $i \le j$ and $s \le t$. Given n, let $k = k(n)$ be the unique integer that satisfies $km \le n < (k+1)m$. Let $\varepsilon > 0$. Then for large enough n,

$$
\begin{aligned}
\xi\left((k+1)mr, (k+1)m(t - \varepsilon)\right) &\le \xi([nr], nt) \\
&\le \xi\left(kmr, km(t + \varepsilon)\right),
\end{aligned}
$$

from which, upon dividing by n and passing to the limit,

$$
\begin{aligned}
\frac{1}{m}\gamma\left(mr, m(t - \varepsilon)\right) &\le \liminf_{n \to \infty} n^{-1} \xi([nr], nt) \\
&\le \limsup_{n \to \infty} n^{-1} \xi([nr], nt) \\
&\le \frac{1}{m}\gamma\left(mr, m(t + \varepsilon)\right).
\end{aligned}
$$

These inequalities cannot be taken literally since the limits have to be interpreted in the sense of convergence in probability. But the point should be clear: By the continuity (33) and the definition (34), the limit (35) is valid.

Next one checks that the homogeneity extends to rationals:

$$\gamma(rx, rt) = r\gamma(x, t) \tag{36}$$

for rational $x \leq 0$, rational $r \geq 0$, and all $t > 0$. From (29) follows subadditivity

$$\gamma(x + y, s + t) \leq \gamma(x, s) + \gamma(y, t), \tag{37}$$

first for integers x, y and then by (34) for rational x, y. (36) and (37) together imply convexity, and also continuity in the x-variable, for rational x.

The final extension of γ is

$$\gamma(x, t) = \sup\{\gamma(r, t) : r \in \mathbf{Q}, r < x\}. \tag{38}$$

By the continuity in the x-variable, for rational x this definition gives the old value $\gamma(x, t)$, so it is a sensible extension. Again, one checks that homogeneity (36) and subadditivity (37) hold, this time for all r, x, y. Convexity follows, and convexity implies continuity in the open quadrant $\{x < 0, t > 0\}$ (Theorem 10.1 in [6]). Continuity up to the boundary requires separate arguments. And again, one checks that the limit (35) holds also for irrational r. This follows as it did for rational r, by the monotonicity of the process and the continuity of the limit.

Finally, define $g(x) = \gamma(x, 1)$. By homogeneity $\gamma(x, t) = tg(x/t)$. ∎

We return to the proof of Theorem 1. Fix (x, t). From the coupling (19) we get

$$n^{-1}\sigma_n([nx], nt) = \inf_{i:i\geq[nx]} \left\{ n^{-1}\sigma_n(i, 0) + n^{-1}\xi_n^i([nx] - i, nt) \right\}. \tag{39}$$

By the hypothesis of Theorem 1 and by Lemma 3, for $i = [ny]$ the random variable inside the braces converges to the quantity inside the braces in (23). Consequently we get

$$\lim_{n\to\infty} P_n \left(n^{-1}\sigma_n([nx], nt) \leq u(x, t) + \varepsilon \right) = 1 \tag{40}$$

for any $\varepsilon > 0$.

For the converse we need some further estimation. The first step is to restrict the variable i in (39) to a range of order n. This is achieved by the next lemma. Define

$$w_n(z) = \min_{i:[nx]\leq i\leq[nz]} \left\{ \sigma_n(i, 0) + \xi_n^i([nx] - i, nt) \right\}. \tag{41}$$

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Lemma 4 *For large enough $z > x$, there exists a constant $C = C(z) > 0$ such that*

$$P_n\left(\sigma_n([nx], nt) \neq w_n(z)\right) \leq e^{-Cn} \tag{42}$$

for all large enough n.

Proof. Suppose particle $\xi_n^{[nz]}([nx] - [nz])$ has not moved by time nt, in other words that $\xi_n^{[nz]}([nx] - [nz], nt) = [nx] - [nz]$. Let $i > [nz]$. By the exclusion rule for σ_n and because

$$\xi_n^i\left([nx] - i, nt\right) \geq \xi_n^i\left([nx] - i, 0\right) = [nx] - i \,,$$

we have

$$
\begin{aligned}
&\sigma_n(i, 0) + \xi_n^i\left([nx] - i, nt\right)\\
\geq \quad &\sigma_n\left([nz], 0\right) + i - [nz] + [nx] - i\\
= \quad &\sigma_n\left([nz], 0\right) + [nx] - [nz]\\
= \quad &\sigma_n\left([nz], 0\right) + \xi_n^{[nz]}\left([nx] - [nz], nt\right) \,.
\end{aligned}
$$

The conclusion is that the terms $i > [nz]$ cannot contribute to the infimum in (39).

Thus to prove the lemma, it suffices to show that for large enough z, the probability that $\xi_n^{[nz]}([nx] - [nz])$ has moved by time nt is at most e^{-Cn}. This is clear again on account of the exclusion and no-passing rules: The random time when $\xi_n^{[nz]}([nx] - [nz])$ first moves has the distribution of a sum of $[nz] - [nx] + 1$ i.i.d. exponential random variables with rate B_0, because after $\xi(j)$ first moves, $\xi(j - 1)$ waits an $\mathrm{Exp}(B_0)$-distributed independent time to make its first jump, and these times are added up over $j = 0, -1, -2, \ldots$, $[nx] - [nz]$. It is sufficient to choose z so that $(z - x)B_0^{-1} > t + \varepsilon$ for some $\varepsilon > 0$. ∎

Fix z so that (42) holds, and let $\varepsilon > 0$. Pick a partition

$$x = y_0 < y_1 < y_2 < \cdots < y_m = z$$

so that

$$\left| tg\left(\frac{x - y_{k+1}}{t}\right) - tg\left(\frac{x - y_k}{t}\right) \right| \leq \frac{\varepsilon}{4} \tag{43}$$

for $k = 0, 1, \ldots, m - 1$. If we could restrict the minimum in (41) to the $m + 1$ points $i = [ny_k]$, then passing to the limit inside or outside the minimum (now

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over a fixed finite number of random variables) would make no difference. To control the error of this simplification, we need to control what happens inside the braces in (41) for $[ny_k] < i < [ny_{k+1}]$. This can be done with couplings:

Lemma 5 *For all n and $i \le j_0 \le j_1$, the inequality $\xi_n^{j_0}(i-j_0, t) \ge \xi_n^{j_1}(i-j_1, t)$ holds almost surely for all times $t \ge 0$.*

Proof. Fix n and $j_0 \le j_1$. Consider the processes

$$\sigma'(i, t) = \xi_n^{j_1}(i - j_1, t) \text{ and } \sigma''(i, t) = \xi_n^{j_0}(i - j_0, t), \ i \in \mathbf{Z}.$$

Both $\sigma'(i)$ and $\sigma''(i)$ read their jump commands from $\mathcal{D}_i$. (Recall the general rule from Section 4: particle $\xi^j(i)$ reads jump commands from $\mathcal{D}_{i+j}$.) Initially

$$\sigma'(i, 0) = \begin{cases} \infty & \text{if } i > j_1, \\ i - j_1 & \text{if } i \le j_1, \end{cases} \tag{44}$$

and

$$\sigma''(i, 0) = \begin{cases} \infty & \text{if } i > j_0, \\ i - j_0 & \text{if } i \le j_0. \end{cases} \tag{45}$$

By the assumption $j_0 \le j_1$, $\sigma'(i, 0) \le \sigma''(i, 0)$, and consequently by Lemma 1, the inequality $\sigma'(i, t) \le \sigma''(i, t)$ holds for all times $t \ge 0$. ∎

Apply Lemma 5 with $i = [nx]$, $j_0 = i$ between $[ny_k]$ and $[ny_{k+1}]$, and $j_1 = [ny_{k+1}]$, to bound $w_n(z)$ from below by

$$
\begin{aligned}
w_n(z) &= \min_{0 \le k < m} \min_{i:[ny_k] \le i \le [ny_{k+1}]} \left\{ \sigma_n(i, 0) + \xi_n^i([nx] - i, nt) \right\} \\
&\ge \min_{0 \le k < m} \left\{ \sigma_n([ny_k], 0) + \xi_n^{[ny_{k+1}]}([nx] - [ny_{k+1}], nt) \right\}.
\end{aligned}
$$

Multiplied through by n^{-1}, the last line converges in probability by assumption (20) and Lemma 3. Thus with probability tending to 1 as $n \to \infty$,

$$
\begin{aligned}
n^{-1} w_n(z) &\ge \min_{0 \le k < m} \left\{ u_0(y_k) + tg\left(\frac{x - y_{k+1}}{t}\right) \right\} - \frac{\varepsilon}{4} \\
&\ge \min_{0 \le k < m} \left\{ u_0(y_k) + tg\left(\frac{x - y_k}{t}\right) \right\} - \frac{\varepsilon}{2} \\
&\ge \inf_{y:y \ge x} \left\{ u_0(y) + tg\left(\frac{x - y}{t}\right) \right\} - \frac{\varepsilon}{2} \\
&= u(x, t) - \frac{\varepsilon}{2}.
\end{aligned}
$$

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This together with Lemma 4 gives

$$\lim_{n\to\infty} P_n\left(n^{-1}\sigma([nx], nt) \geq u(x,t) - \varepsilon\right) = 1 \qquad (46)$$

for any $\varepsilon > 0$. This completes the proof of Theorem 1.

6 Comments and extensions

6.1 Almost sure convergence in Theorem 1

It may be possible to upgrade the convergence in Theorem 1 to almost sure convergence by developing large deviation estimates for the convergence in Lemma 3. Such results can be found in [9] and [10].

6.2 The role of invariant distributions

Theorem 1 does not explicitly identify the function g or the macroscopic velocity f. This can be done when information about steady states is available. Consider the special case of the totally asymmetric simple exclusion process (TASEP) where $\beta_1 = 1$ and $\beta_k = 0$ for $k > 1$. Then it is known that if the interparticle distances are geometrically distributed with mean $v \in [1, \infty)$,

$$P\left(\sigma(i+1) - \sigma(i) = m\right) = \left(1 - v^{-1}\right)^{m-1} v^{-1}, \quad m = 1, 2, 3, \ldots, \qquad (47)$$

an individual particle's jumps obey a Poisson process of rate $1 - v^{-1}$. From this information the function g can be calculated with the help of Theorem 1, as we next show.

Let each $\sigma_n = \sigma$ be a steady-state process with interparticle distances (47), and such that initially $\sigma(0,0) = 0$. Then Theorem 1 is valid for $u_0(y) = vy$. Take $x = 0$ and $t = 1$ in (23) to get

$$u(0,1) = \inf_{y \geq 0}\{vy + g(-y)\} = -g^*(v), \qquad (48)$$

where g^* denotes the convex conjugate of g. Note: For the purposes of convex duality, functions should be lower semicontinuous and convex on all of $\mathbf{R}$. (See [6].) The limit of Lemma 3 defines g on $(-\infty, 0]$, and we extend g to $\mathbf{R}$ by defining $g(x) = \infty$ for $x > 0$. Then (48) is the same as

$$g^*(v) = \sup_{x}\{xv - g(x)\}. \qquad (49)$$

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Furthermore, for $v < 1$ we obtain $g^*(v) = \infty$ directly from (49), by the observation that the slope of g is always at least 1. This is a consequence of the exclusion rule (1) and the limit (25).

Since $\sigma(0, n)$ is a Poisson$((1 - v^{-1})n)$-variable, we can calculate explicitly that

$$u(0, 1) = \lim_{n \to \infty} n^{-1}\sigma(0, n) = 1 - v^{-1}.$$

Thus $g^*(v) = v^{-1} - 1$ for $v \geq 1$. From this we obtain g by performing another convex conjugation: For $x \leq 0$,

$$
\begin{aligned}
g(x) &= g^{**}(x) \\
&= \sup_{v}\{xv - g^*(v)\} \\
&= \sup_{v \geq 1}\{xv - v^{-1} + 1\} \\
&= 1 - 2\sqrt{-x}.
\end{aligned}
$$

The tagged particle velocity is the function $f(v) = 1 - v^{-1}$ obtained above. The relationship of the mean interparticle distance v to the particle density ρ is $\rho = v^{-1}$. As a function of density the velocity is $v(\rho) = f(\rho^{-1}) = 1 - \rho$. From this we get the well-known macroscopic *particle current* of TASEP: $\rho v(\rho) = \rho(1 - \rho)$.

TASEP was the process studied by Rost in 1981 [7] in one of the seminal papers on hydrodynamic limits for asymmetric processes. With an approach completely different from ours he investigated the process ξ and proved a limit theorem related to Lemma 3. The use of subadditivity is a common feature of both approaches.

A thorough treatment of TASEP with the method of this paper appears in [8]. In particular, when the invariant distributions are available for explicit calculations, in addition to the law of large numbers it is possible to obtain a (one-sided) explicit large deviation rate function for the tagged particle in a nonequilibrium process.

6.3   More general jump rates

We could admit the following additional feature to the process: Not only is $\sigma(i)$ blocked by the particle in front of him, but he is also held back by the

particle behind him, in this particular sense: For some constant $L \geq 1$,

$$\sigma(i) \leq \sigma(i-1) + L \text{ for all } i \in \mathbf{Z}. \tag{50}$$

The proof of Theorem 1 works with suitable modifications. The initial configurations of the ζ^j-processes have to be changed to $\zeta^j(i) = \sigma(j,0) + i$ for $i \leq 0$, and $\zeta^j(i) = \sigma(j,0) + Li$ for $i > 0$. This feature appears in [9].

Another generalization that this approach can handle is *random rates*. The reader may have noticed that very little depended on how the point processes $\mathcal{D}_i^k$ were actually produced, as long as they had some desirable properties such as those listed in assumption (5). In particular, the rates of the processes could vary so that the rate $\beta_{k,i}$ of $\mathcal{D}_i^k$ is a random variable whose realization is fixed. If this randomness is suitably systematic (i.i.d. for example) one would expect that some form of the subadditive ergodicity needed for Lemma 3 works again. Examples appear in [9] and [10].

However, so far it has not been possible to admit a more complicated dependence of the rate of jumping on the distance to the nearest particles. By a more complicated dependence we mean a general function $\beta(k,l,r)$ such that jumps of size k are attempted at rate $\beta(k,l,r)$ when there are l and r empty sites to the left and right. In the model of this paper this function is a step function: For a fixed k, $\beta(k,l,r)$ takes only two values depending on whether $k < r$ or $k \geq r$. What appears to fail for more general rates is the proof of Lemma 2.

Other shortcomings are that the method has been able to treat only totally asymmetric processes, and processes without passing. A more general asymmetric process would allow jumps both left and right, but with a higher rate for jumping to the right.

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